Tentamen Metrische Ruimten, 23/08/04

1. Let B be a totally bounded subset of a metric space M. Show that B is bounded. Give an example of a bounded metric space which is not totally bounded.

2. Let $f_n \in C[0,1]$ be defined by

$$f_n(x) = \frac{13x^n}{7n}, \quad x \in [0,1], n \in \mathbb{N}.$$

   (a) Formulate the Theorem of Arzela-Ascoli.

   (b) Let $F := \{f_n : n \in \mathbb{N}\}$. What is $cl(F)$ in $C[0,1]$ with respect to the d_∞-metric?

   (c) Is $cl(F)$ compact in $C[0,1]$?

3. Let $\mathfrak{T}$ consist of all subsets U of the set of real numbers $\mathbb{R}$ such that $\mathbb{R} \setminus U$ is finite, together with the empty set $\emptyset$.

   (a) Show that $(\mathbb{R}, \mathfrak{T})$ is a topological space.

   (b) Is $(\mathbb{R}, \mathfrak{T})$ compact?

   (c) Is $(\mathbb{R}, \mathfrak{T})$ Hausdorff?

   (d) Is $(\mathbb{R}, \mathfrak{T})$ connected?

   (e) Can $\mathfrak{T}$ be generated by a (non-Euclidean) metric defined on $\mathbb{R}$?

   Justify the answers!

4. Determine the closure and the boundary of each of the following subsets of $\mathbb{R}$ with the usual Euclidean metric. Which of these sets are dense or nowhere dense in $\mathbb{R}$? ($\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ the set of integers.)

   (a) $\mathbb{R}$;

   (b) $\mathbb{Q} \cap [-1, 2)$;

   (c) $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$;

   (d) $\mathbb{R} \setminus \mathbb{Q}$;

   (e) $\mathbb{R} \setminus \mathbb{Z}$;

   (f) $\mathbb{Z}$.